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The First Integral Method for the Two-dimensional Incompressible Navier-Stokes Equations

Ammar Abd Al-Hussein[1*]

[1]Department of Communication Engineering, Iraq University College, Basrah, Iraq.

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper, we deal with the first integral method to find exact solutions for The Two-Dimensional Incompressible Navier-Stokes equations. This method is an algebraic direct method used division theorem to find the first integral through polynomial and use traveling wave solution to transform the partial differential equation into the ordinary differential equation. We get different exact solutions through the use of this method and these solutions are either of the formula of exponential, hyperbolic or trigonometric functions.

Keywords: *Commutative algebra theory; first integral method; the incompressible two-dimensional Navier-Stokes equations.*

1 Introduction

Still, differential equations are used in understanding the physical sciences, engineering and vital addition to its contribution to the study of mathematical analysis and extended their use in economic sciences. Most of the relations and laws that bind the variables of the issue of physical or engineering appear in differential equations and developed differential equations and the increased importance in all fields of science and its

**Corresponding author: E-mail: ammaralsalih88@gmail.com;*

applications. In order to understand these nonlinear phenomena, which described by differential equations, most researchers paid attention to study them. The investigation of the traveling wave solution of a nonlinear equation is significance in physics and applied sciences such as fluid dynamics, solid state physics, mechanics, biology and mathematical finance. There is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions. The researchers worked to find a solution to the nonlinear differential equations, or at least know a lot of properties of this solution. While there is no existing general theory for nonlinear partial differential equations, many special cases have yielded to appropriate changes of variable. Certain special form solutions may depend only on a single combination of variables such as traveling wave variables. Also using numerical methods to solve differential equations that are difficult to find the exact solution since the solution could not be found in all equations which are very important to the computation of the solution of differential equations to obtain the approximate solutions to these equations. Various methods have been utilized to explore different kinds of solutions of physical models. In recent years, quite few methods for obtaining the analytical solution of PDEs which are using traveling wave solution have been proposed such as tanh-sech method [1,2], extended tanh method [3,4], the generalized hyperbolic function method [5], sine-cosine method [6], F-expansion method [7,8], the transformed rational function method [9] and many other methods. The first integral method is one of the most effective methods to find the exact solutions for some important PDEs. This method used traveling wave solution to transform the PDE into an ODE. Feng [10] proposes a new approach which is currently called the first integral method to study the compound Burgers'-KdV equation and Burgers'-KdV equation and this method based on the ring theory of commutative algebra. Many researchers used this method to obtain new exact solutions to PDEs such as [11-15].

2 Basic Ideas of the First Integral Method

The basic idea of the first integral method is simple and we will review the fundamental concepts as follows:

Let us consider the nonlinear system of PDEs with independent variables x and t and dependent variables u and v:

$$\left.\begin{aligned} R_1(u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx}, \dots) = 0 \\ R_2(u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx}, \dots) = 0 \end{aligned}\right\} \tag{1}$$

Applying the traveling wave solution transformation $u(x,t) = f(\xi)$ and $v(x,t) = g(\xi)$, where $\xi = x - ct$, now, by using the chain rule we get:

$$\frac{\partial}{\partial t}(.) = -c\frac{d}{d\xi}(.), \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.), \frac{\partial^2}{\partial x^2}(.) = \frac{d^2}{d\xi^2}(.), \frac{\partial^2}{\partial t^2}(.) = c^2\frac{d^2}{d\xi^2}(.) \dots \tag{2}$$

Using (2) into (1) to transfer the PDEs into ODEs as follows:

$$\left.\begin{aligned} T_1(f, g, f', g', \dots) = 0 \\ T_2(f, g, f', g', \dots) = 0 \end{aligned}\right\} \tag{3}$$

where the prime denotes the derivative with respect to the same variable ξ.

Make this system a single equation with one dependent variable of the second order, using the integration, we have the equation as follows:

$$H(f, f', f'') = 0 \tag{4}$$

We define new independent variables:

$$X(\xi) = f(\xi), \quad Y(\xi) = f'(\xi) \tag{5}$$

This leads to a system of ODEs:

$$\left.\begin{aligned} X'(\xi) &= Y(\xi), \\ Y'(\xi) &= F(X(\xi), Y(\xi)) \end{aligned}\right\} \tag{6}$$

Now, the division theorem is adapted to obtain one first integral to (6), which reduces (4) to a first-order integrable ODE. Finally, an exact solution to (1) be established, through solving the resulting first-order integrable differential equation.

Division Theorem: Suppose that $P(w, z)$ and $Q(w, z)$ are polynomials of two variables w and z in complex domain $C[w, z]$ and $P(w, z)$ is irreducible in $C[w, z]$. If $Q(w, z)$ vanishes at all zero points of $P(w, z)$, then there exists a polynomial $G(w, z)$ in $C[w, z]$ such that:

$$Q(w, z) = P(w, z)G(w, z)$$

3 The Incompressible Two-Dimensional Navier-Stokes Equations

The Incompressible Navier-Stokes equations are the fundamental partial differentials equations that describe the flow of incompressible fluids and these equations are one of the most useful sets of equations because they can be used to describe many different engineering problems. Many researchers seek the solution of the Incompressible Navier-Stokes equations such as [16,17]. The Incompressible Navier-Stokes equations, named after Claude-Louis Navier and George Gabriel Stokes describe the motion of fluid, i.e. liquids.

The Incompressible two-dimensional Navier-Stokes equations [16,17] are considered as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{7a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{7b}$$

$$u_x + v_y = 0 \tag{7c}$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2 - 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \tag{7d}$$

where $x, y \in \Omega$ such that $\Omega = [0, a] \times [0, b]$ is the bounded domain of the cavity, Re is the Reynolds number and $t \geq 0$ is the time. The stream function $\Psi(x, y, t)$ is defined for two-dimensional flows; the partial derivatives of the stream function are linked with the velocity components through the relation:

$$\frac{\partial \Psi}{\partial x} = -v, \qquad \frac{\partial \Psi}{\partial y} = u \tag{8}$$

The equation $(7a)$ is differentiated with respect to y and the second equation $(7b)$ is differentiated with respect to x. Then, equation $(7a)$ is subtracted from the equation $(7b)$ one, so that the pressure is eliminated.

Substituting the definition of vorticity $w = v_x - u_y$ yields the vorticity transport equation as:

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} - \frac{1}{Re}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{9}$$

Substituting (8)in (9) we get:

$$w_t + \Psi_y w_x - \Psi_x w_y = \frac{1}{Re}\left(w_{xx} + w_{yy}\right) \tag{10}$$

Now we seek the exact solution to equation (10) by applying the first integral method as follows: First, we suppose that:

$$\Psi(x, y, t) = \frac{-2}{(Re)^2 c^2}(\alpha + w(x, y, t)) \tag{11}$$

To solve equation (10) we use the traveling wave solution as follows:

$$w(x, y, t) = f(\xi), \quad \Psi(x, y, t) = g(\xi) \qquad \xi = x + y + ct \tag{12}$$

where c is a constant. By using chain rule equation (10) transform the partial differential equation into ordinary differential equation as follows:

$$f''(\xi) = \frac{Rec}{2} f'(\xi) \quad , \tag{13}$$

and equation (11) becomes:

$$g(\xi) = \frac{-2}{(Re)^2 c^2}\left(\alpha + f(\xi)\right) \tag{14}$$

Hence, we define new independent variables $X(\xi),\ Y(\xi)$ as follows:

$$X'(\xi) = Y \tag{15a}$$

$$Y'(\xi) = \frac{Rec}{2} Y(\xi) \quad , \tag{15b}$$

by the same way in the last examples:

$$p[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X) Y^i = 0 \quad , \tag{16}$$

where $a_i(x)$ $(i = 0,1, \dots, m)$ are polynomials of X and $a_m(X) \neq 0$ and:

$$\frac{dp}{d\xi} = \frac{\partial p}{\partial X}\frac{dX}{d\xi} + \frac{\partial p}{\partial Y}\frac{dY}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X) Y^i \tag{17}$$

We start with two different cases $m = 1$, and $m = 2$ in equation (17).

Case 1: Let $m = 1$, then we obtains:

$$\frac{dp}{d\xi} = \frac{\partial p}{\partial X}\frac{dX}{d\xi} + \frac{\partial p}{\partial Y}\frac{dY}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{1} a_i(X) \tag{18}$$

By comparing the coefficients of Y^i $(i = 2,1,0)$ on both sides of equation (18) we get:

$$a_1'(X) = a_1(X) h(X) \tag{19a}$$

$$a_0^{'}(X) = a_0(X)h(X) + a_1(X)g(X) - \frac{Rec}{2}a_1(X) \tag{19b}$$

$$a_0(X)g(X) = 0 \tag{19c}$$

From $(19a)$ we get $a_1(X)$ is constant and $h(X) = 0$, then we take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$ we conclude that deg $g(X) = 0$ only, and then we suppose that $g(X) = A_1$.

Substituting $g(X)$ and $a_1(X)$ in $(19b)$ then we can find $a_0(X)$:

$$a_0(X) = \left(A_1 - \frac{Rec}{2}\right)X + B_0 \tag{20}$$

where B_0 is an arbitrary integration constants. Substituting $a_0(X)$ and $g(X)$ in $(19c)$ we get:

$$A_1\left(A_1 - \frac{Rec}{2}\right)X + A_1B_0 = 0$$

Setting the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations:

$$\left.\begin{aligned} A_1B_0 &= 0 \\ A_1\left(A_1 - \frac{Rec}{2}\right) &= 0 \end{aligned}\right\} \tag{21}$$

Solving the algebraic equations (21), we obtain:

$$A_1 = 0 \tag{22}$$

Using the condition (22) to (17), we obtain:

$$Y(\xi) = \frac{Rec}{2}X(\xi) - B_0 \tag{23}$$

Combining (23) with $(15b)$, we obtain:

$$\frac{dX}{\frac{Rec}{2}X(\xi) - B_0} = d\xi$$

By integrating both sides we get:

$$X(\xi) = \frac{2B_0}{Rec} + \frac{2}{Rec}e^{\frac{Rec}{2}(\xi+\xi_0)}$$

From $(15a)$ we obtain the exact solution to (13) as follows:

$$f(\xi) = \frac{2B_0}{Rec} + \frac{2}{Rec}e^{\frac{Rec}{2}(\xi+\xi_0)} \tag{24}$$

Substituting (24) in (14), we get:

$$g(\xi) = \frac{-2}{(Re)^2c^2}\left[\alpha + \frac{2B_0}{Rec} + \frac{2}{Rec}e^{\frac{Rec}{2}(\xi+\xi_0)}\right]$$

where ξ_0 is an arbitrary constant. From (12) the exact solutions to the equation (10) can be written as:

$$w(x,y,t)=\frac{2B_0}{Rec}+\frac{2}{Rec}e^{\frac{Rec}{2}(x+y+ct+\xi_0)} \tag{25}$$

$$\Psi(x,y,t)=\frac{-2}{(Re)^2c^2}\left[\alpha+\frac{2B_0}{Rec}+\frac{2}{Rec}e^{\frac{Rec}{2}(x+y+ct+\xi_0)}\right] \tag{26}$$

Case 2: Let $m=2$ in (17), due to the division theorem, we get:

$$\frac{dp}{d\xi}=\frac{\partial p}{\partial X}\frac{dX}{d\xi}+\frac{\partial p}{\partial Y}\frac{dY}{d\xi}=(g(X)+h(X)Y)\sum_{i=0}^{2}a_i(X)\,Y^i \tag{27}$$

By comparing the coefficients of $Y^i(i=3,2,1,0)$ on both sides of equation (27) we get:

$$a_2'(X)=a_2(X)h(X) \tag{28a}$$

$$a_1'(X)=a_1(X)h(X)+a_2(X)g(X)-Reca_2(X) \tag{28b}$$

$$a_0'(X)=a_1(X)g(X)+a_0(X)h(X)-\frac{Rec}{2}a_1(X) \tag{28c}$$

$$a_0(X)g(X)=0 \tag{28d}$$

From $(28a)$, we deduce that $h(X)=0$ and $a_2(X)=1$.

Balancing the degrees of $g(X)$, $a_1(X)$ and $a_0(X)$ we conclude that $\deg g(X)=0$ only. Then we suppose that $g(X)=A_1$.

Substituting $g(X), a_2(X)$ in $(28b)$, then we find $a_1(X)$:

$$a_1(X)=(A_1-Rec)X+B_0 \tag{29}$$

Substituting $g(X), a_2(X)$ and $a_1(X)$ in $(28c)$, then we can find $a_0(X)$:

$$a_0(X)=A_0+\left(A_1B_0-\frac{RecB_0}{2}\right)X+\left(\frac{A_1^2}{2}-\frac{3A_1Rec}{4}+\frac{(Re)^2c^2}{4}\right)X^2 \tag{30}$$

where A_0 and B_0 are integration constants. Substituting $a_0(X)$ and $g(X)$ in $(28d)$ we get:

$$A_1A_0+A_1\left(A_1B_0-\frac{RecB_0}{2}\right)X+A_1\left(\frac{A_1^2}{2}-\frac{3A_1Rec}{4}+\frac{(Re)^2c^2}{4}\right)X^2=0$$

Setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations:

$$A_1A_0=0 \tag{31a}$$

$$A_1\left(A_1B_0-\frac{ReB_0c}{2}\right)=0 \tag{31b}$$

$$A_1\left(A_1^2-\frac{3A_1Rec}{2}+\frac{(Re)^2c^2}{2}\right)=0 \tag{31c}$$

Solving the algebraic (31) equations, we obtain:

$$A_1 = 0 \tag{32}$$

Using the condition (32) to (17), we obtain:

$$Y(\xi) = \frac{RecX(\xi) - B_0 + \sqrt{B_0^2 - 4A_0}}{2} \tag{33a}$$

$$Y(\xi) = \frac{RecX(\xi) - B_0 - \sqrt{B_0^2 - 4A_0}}{2} \tag{33b}$$

Combining $(33a)$ with $(15b)$, we obtain:

$$\frac{2\,dX}{RecX(\xi) - B_0 + \sqrt{B_0^2 - 4A_0}} = d\xi$$

By integrating both sides we get:

$$X(\xi) = \frac{B_0 - \sqrt{B_0^2 - 4A_0}}{Rec} + \frac{2e^{\frac{Rec}{2}(\xi+\xi_0)}}{Rec}$$

From $(15a)$ we obtain the exact solution to (13) as follows:

$$f(\xi) = \frac{B_0 - \sqrt{B_0^2 - 4A_0}}{Rec} + \frac{2e^{\frac{Rec}{2}(\xi+\xi_0)}}{Rec} \tag{34}$$

Substituting (34) into (14), we get:

$$g(\xi) = \frac{-2}{(Re)^2c^2}(\alpha + \frac{B_0 - \sqrt{B_0^2 - 4A_0}}{Rec} + \frac{2e^{\frac{Rec}{2}(\xi+\xi_0)}}{Rec})$$

where ξ_0 is an arbitrary constant. From (12) the exact solutions to equation (10) can be written as:

$$w(x,y,t) = \frac{B_0 - \sqrt{B_0^2 - 4A_0}}{Rec} + \frac{2e^{\frac{Rec}{2}(x+y+ct+\xi_0)}}{Rec} \tag{35}$$

$$\Psi(x,y,t) = \frac{-2}{(Re)^2c^2}\left(\alpha + \frac{B_0 - \sqrt{B_0^2 - 4A_0}}{Rec} + \frac{2e^{\frac{Rec}{2}(x+y+ct+\xi_0)}}{Rec}\right) \tag{36}$$

Combining $(33b)$ with $(15b)$, we obtain:

$$\frac{2\,dX}{RecX(\xi) - (B_0 + \sqrt{B_0^2 - 4A_0})} = d\xi$$

By integrating both sides we get:

$$X(\xi) = \frac{B_0 + \sqrt{B_0^2 - 4A_0}}{Rec} + \frac{2e^{\frac{Rec}{2}(\xi+\xi_0)}}{Rec}$$

From $(15a)$ we obtain the exact solution to (13) as follows:

$$f(\xi) = \frac{B_0 + \sqrt{B_0^2 - 4A_0}}{Rec} + \frac{2e^{\frac{Rec}{2}(\xi+\xi_0)}}{Rec} \tag{37}$$

Substituting (37) in (14), we get:

$$g(\xi) = \frac{-2}{(Re)^2c^2}\left(\alpha + \frac{B_0+\sqrt{B_0^2-4A_0}}{Rec} + \frac{2e^{\frac{Rec}{2}(\xi+\xi_0)}}{Rec}\right)$$

where ξ_0 is an arbitrary constant. From (12) the exact solutions to equation (10) can be written as:

$$w(x,y,t) = \frac{B_0 + \sqrt{B_0^2 - 4A_0}}{Rec} + \frac{2e^{\frac{Rec}{2}(x+y+ct+\xi_0)}}{Rec} \tag{38}$$

$$\Psi(x,y,t) = \frac{-2}{(Re)^2c^2}\left(\alpha + \frac{B_0+\sqrt{B_0^2-4A_0}}{Rec} + \frac{2e^{\frac{Rec}{2}(x+y+ct+\xi_0)}}{Rec}\right) \tag{39}$$

Then, we get:

$$u(x,y,t) = \frac{-2}{(Re)^2c^2}e^{\frac{Rec}{2}(x+y+ct+\xi_0)} \tag{40}$$

$$v(x,y,t) = \frac{2}{(Re)^2c^2}e^{\frac{Rec}{2}(x+y+ct+\xi_0)} \tag{41}$$

These solutions are new exact solutions and very important in different phenomena.

4 Conclusions

The first integral method is an accurate, effective, not complicated when it is applied and efficient to using for solving PDEs where we get various exact solutions for any considering PDEs and its reliable application does not require complex and tedious calculations and uses traveling wave solutions and division theory provides the first integral in polynomial form through which we get more exact solutions of the PDEs.

Competing Interests

Author has declared that no competing interests exist.

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