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Extending the Theory of k-Fibonacci and k-Lucas Numbers

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Short Communication

Abstract

The a:k:m-Fibonacci sequences $F_{a:k:m,n}\begin{cases} f_{a:k:m,n+2} = kf_{a:k:m,n+1} + amf_{a:k:m,n}, a,k,m,n \geq 1 \\ f_{a:k:m,1} = 1, f_{a:k:m,2} = k \end{cases}$ and the a:k:m-Lucas sequences $L_{a:k:m,n}\begin{cases} l_{a:k:m,n+2} = kl_{a:k:m,n+1} + aml_{a:k:m,n}, a,k,m,n \geq 1 \\ l_{a:k:m,1} = k, l_{a:k:m,2} = k^2 + 2am \end{cases}$ are introduced. The well-known k-Fibonacci and k-Lucas sequences become a particular case (a=m=1). One might be interested to meet the equally famous Jacobsthal sequence at $a = 2, k = m = 1$. Our brief results capture the most important properties relating to the assemblage mechanics of these sequences.

Keywords: a:k:m-Fibonacci numbers; a:k:m-Lucas numbers; Jacobsthal numbers; k-Fibonacci numbers; k-Lucas numbers; metallic means.

1 Introduction

The sequence of numbers

$$F_n = 1,1,2,3,5,8,\ldots \quad (1.1)$$

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well-known as the Fibonacci numbers in the literature [1-32], is arithmetically generated by the recurrence relation

$$f_{n+2} = f_{n+1} + f_n, n \geq 1 \tag{1.2}$$

with initial conditions

$$f_1 = f_2 = 1 \tag{1.3}$$

The k-Fibonacci numbers introduced by Falcon and Plaza [1] are defined by

$$F_{k,n} \begin{cases} f_{k,n+2} = kf_{k,n+1} + f_{k,n}, \quad n, k \geq 1 \\ f_{k,1} = 1, f_{k,2} = k \end{cases} \tag{1.4}$$

In this communication we introduce the a:k:m-Fibonacci numbers defined by

$$F_{a:k:m,n} \begin{cases} f_{a:k:m,n+2} = kf_{a:k:m,n+1} + amf_{a:k:m,n}, a, k, m, n \geq 1 \\ f_{a:k:m,1} = 1, f_{a:k:m,2} = k \end{cases} \tag{1.5}$$

and the related a:k:m-Lucas numbers defined by

$$L_{a:k:m,n} \begin{cases} l_{a:k:m,n+2} = kl_{a:k:m,n+1} + aml_{a:k:m,n}, a, k, m, n \geq 1 \\ l_{a:k:m,1} = k, l_{a:k:m,2} = k^2 + 2am \end{cases} \tag{1.6}$$

based on the positive solution of the quadratic equation

$$ax^2 - kx - m = 0 \tag{1.7}$$

Let's denote this solution $\Omega_a^{k:m}$. We have that

$$\Omega_a^{k:m} = \frac{k + \sqrt{k^2 + 4am}}{2a} \tag{1.8}$$

Our very brief results are intended to show that the basic properties of the sequence (1.1) and the Lucas sequence are retained in all the a:k:m-Fibonacci and a:k:m-Lucas sequences respectively. One might be interested to learn that, for instance, the well-known Jacobsthal numbers

$$F_{2:1:1,n} = F_{1:1:2,n} = 1,1,3,5,11, \ldots \tag{1.9}$$

are in fact employing the same concept as the Fibonacci, Pell, etc. numbers.

2 Results

Theorem 2.1

$$a^{n-1}(\Omega_a^{k:m})^n - f_{a:k:m,n}\Omega_a^{k:m} = mf_{a:k:m,n-1}, n \geq 1 \tag{2.1}$$

Proof

By induction. Base case: $n = 1$,

$$\Omega_a^{k:m} - \Omega_a^{k:m} = 0 = mf_{a:k:m,0}$$

Inductive Hypothesis:

$$a^{i-1}(\Omega_a^{k:m})^i - f_{a:k:m,i}\Omega_a^{k:m} = mf_{a:k:m,i-1}, i \geq 1 \tag{2.2}$$

Inductive Conclusion:

$$a^{i}(\Omega_a^{k:m})^{i+1} - f_{a:k:m,i+1}\Omega_a^{k:m} = mf_{a:k:m,i}, i \geq 1 \tag{2.3}$$

We have that

$$\begin{aligned}
&a^{i}(\Omega_a^{k:m})^{i+1} - f_{a:k:m,i+1}\Omega_a^{k:m} \\
&= a^i\left(\Omega_a^{k:m}\right)^i\left(\Omega_a^{k:m}\right) - kf_{a:k:m,i}\Omega_a^{k:m} - amf_{a:k:m,i-1}\Omega_a^{k:m} \\
&= k\left(a^{i-1}\left(\Omega_a^{k:m}\right)^i - f_{a:k:m,i}\Omega_a^{k:m}\right) + \frac{-k+\sqrt{k^2+4am}}{2a}(a^i)(\Omega_a^{k:m})^i - amf_{a:k:m,i-1}\Omega_a^{k:m} \\
&= kmf_{a:k:m,i-1} + m(a^{i-1})(\,\Omega_a^{k:m})^{i-1} - amf_{a:k:m,i-1}\Omega_a^{k:m} \\
&= kmf_{a:k:m,i-1} + ma((a^{i-2})(\,\Omega_a^{k:m})^{i-1} - f_{a:k:m,i-1}\Omega_a^{k:m}) \\
&= kmf_{a:k:m,i-1} + am^2f_{a:k:m,i-2} \\
&= mf_{a:k:m,i}
\end{aligned}$$

Induction is concluded, proof is complete.

Theorem 2.2

$$f_{a:k:m,n+1} = af_{a:k:m,n}\Omega_a^{k:m} + \left(\frac{-m}{\Omega_a^{k:m}}\right)^n, n \geq 1 \tag{2.4}$$

Proof

By induction. Base case: $n = 1$,

$$a\Omega_a^{k:m} + \left(\frac{-m}{\Omega_a^{k:m}}\right)^1 = k = f_{a:k:m,2}$$

Inductive Hypothesis:

$$f_{a:k:m,i+1} = af_{a:k:m,i}\Omega_a^{k:m} + \left(\frac{-m}{\Omega_a^{k:m}}\right)^i, i \geq 1 \tag{2.5}$$

Inductive Conclusion: We prove that

$$f_{a:k:m,i+2} = af_{a:k:m,i+1}\Omega_a^{k:m} + \left(\frac{-m}{\Omega_a^{k:m}}\right)^{i+1}, i \geq 1 \tag{2.6}$$

We obtain

$$\begin{aligned}
&af_{a:k:m,i+1}\Omega_a^{k:m} + \left(\frac{-m}{\Omega_a^{k:m}}\right)^{i+1} \\
&= k\left(af_{a:k:m,i}\Omega_a^{k:m} + \left(\frac{-m}{\Omega_a^{k:m}}\right)^i\right) - \frac{k+\sqrt{k^2+4am}}{2}\left(\frac{-m}{\Omega_a^{k:m}}\right)^i + am^2f_{a:k:m,i-1}\Omega_a^{k:m} \\
&= kf_{a:k:m,i+1} + am(af_{a:k:m,i-1}\Omega_a^{k:m} + \left(\frac{-m}{\Omega_a^{k:m}}\right)^{i-1}) \\
&= kf_{a:k:m,i+1} + amf_{a:k:m,i} \\
&= f_{a:k:m,i+2}
\end{aligned}$$

Having concluded the induction process, proposition is true.

Theorem 2.3

$$l_{a:k:m,n} = a^n(\Omega_a^{k:m})^n + \left(\frac{-m}{\Omega_a^{k:m}}\right)^n, n \geq 1 \tag{2.7}$$

Derivation

Notice that, by definition,

$$l_{a:k:m,n} = amf_{a:k:m,n-1} + f_{a:k:m,n+1} \tag{2.8}$$

From proved equations (2.1) and (2.4) this becomes

$$a^n(\Omega_a^{k:m})^n - f_{a:k:m,n}\Omega_a^{k:m} + af_{a:k:m,n}\Omega_a^{k:m} + \left(\frac{-m}{\Omega_a^{k:m}}\right)^n$$
$$= a^n(\Omega_a^{k:m})^n + \left(\frac{-m}{\Omega_a^{k:m}}\right)^n$$

Theorems 2.1 to 2.3 capture the basic properties of a:k:m-Fibonacci and a:k:m-Lucas sequences relating to assembly mechanics. Without further proof we state Catalan's and d'Ocagne's identities respectively:

$$f_{a:k:m,n}^2 - f_{a:k:m,n+r}f_{a:k:m,n-r} = (-1)^{n-r}(am)^{n-r}f_{a:k:m,r}^2, n, r \geq 1 \tag{2.9}$$

$$f_{a:k:m,r}f_{a:k:m,n+1} - f_{a:k:m,n}f_{a:k:m,r+1} = (-1)^n(am)^n f_{a:k:m,r-n}, n, r \geq 1 \tag{2.10}$$

3 Conclusion

The a:k:m-Fibonacci and a:k:m-Lucas sequences extend not only the theory of k-Fibonacci and k-Lucas numbers but of metallic means [10] also. That we have shown that the Jacobsthal numbers [19] for example employ the same concept as the classic Fibonacci numbers goes a long way in the unification of seemingly disparate ideas and opening new avenues of research.

Competing Interests

Author has declared that no competing interests exist.

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