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Existence of Global Attractor for a Hyperbolic Phase Field System of Caginalp Type with Polynomial Growth Potential

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

Aims/ objectives: We are interested in a hyperbolic phase field system of Caginalp type, parameterized by ϵ for which the solution is a function defined on $(0, T) \times \Omega$. We show the existence of the global attractor for a hyperbolic phase field system of Caginalp type, with homogenous conditions Dirichlet on the boundary, this system is governed by a polynomial growth potential, in a bounded and smooth domain. the hyperbolic phase field system of Caginalp type is based on a thermomecanical theory of deformable continua.
Note that the global attractor is the smallest compact set in the phase space, which is invariant by the semigroup and attracts all bounded sets of initial data, as time goes to infinity. So the global attractor allows to make description of asymptotic behaviour about dynamic system.
Study Design: Propagation study of waves.

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Methodology: To show the existence of the global attractor about the perturbed damped hyperbolic system, with initial conditions and homogenous conditions Dirichlet on the boundary, we proceed by proving the dissipativity and regularity of the semigoup associated to the system, and we then split the semigroup such that we have the sum of two continuous operators, where the first tends uniformly to zero when the time goes to infinity, and the second is regularizing.
Results: We show the existence of global attractor, about a hyperbolic phase field system of Caginalp type, governed by polynomial growth potential.
Conclusion: All the procedures explained in the methodology being demonstrated , we can assert the existence of the smallest compact set of the phase space, invariant by the semigroup and which attracts all the bounded sets of initial data from a some time.

Keywords: The hyperbolic phase field system of Caginalp type; polynomial growth potential; conditions Dirichlet on boundary; dissipativity; global attractor.

2016 Mathematics Subject Classification: Primary: 35D05, 35D10, 35D20, 35D25, 35D30, Secondary: 46F30.

1 Introduction and Setting of the Problem

We recall that the global attractor $\mathcal{A}$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e. $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$) and attracts all bounded sets, of initial data when time goes to infinity. The property of invariance satisfied by the global attractor makes sure of its unicity (when the global attractor exists). It is the smallest closed set which verifies the property of attraction; and it thus appears as a suitable object in view of the study of the asymptotic behaviour of the system. In fact the global attractor is the smallest compact set of the phase space which contains the solution of a dynamic system, when time goes to infinity.

The Caginalp phase field system

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u + f(u) &= \theta, \\ \frac{\partial \theta}{\partial t} - \Delta \theta &= - \frac{\partial u}{\partial t}, \end{aligned}$$

has been proposed in [1] to model phase transition phenomena, such that melting-solidification phenomena, in certain classes of materials. In this context, $u = u(t,x)$ denotes the phase-field or the order parameter, $\theta = \theta(t,x)$ stands for the relative temperature defined as $\theta = \frac{\partial \alpha}{dt}$, where $\alpha = \alpha(t,x)$ is the thermal displacement variable or the primitive of θ and f is the derivative of a double-well potentiel F.

In this paper, we are based on the following system of Caginalp type

$$\epsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial \alpha}{\partial t}, \tag{1.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = - u - \frac{\partial u}{\partial t}, \tag{1.2}$$

with homogenous conditions Dirichlet on the boundary

$$u|_{\partial\Omega} = \alpha|_{\partial\Omega} = 0, \tag{1.3}$$

and initial conditions

$$u|_{t=0} \;=\; u_0,\;\; \frac{\partial u}{\partial t}|_{t=0} \;=\; u_1,\;\; \alpha|_{t=0} \;=\; \alpha_0,\;\; \frac{\partial \alpha}{\partial t}|_{t=0} \;=\; \alpha_1, \tag{1.4}$$

which is the hyperbolic relaxation system of Caginalp phase-field based on the type III (see [4]). Ω is a bounded and smooth domain of class C^2 in $\mathbb{R}^n$ $(1 \le n \le 3)$, $\partial\Omega$ the smooth boundary of Ω, and $\epsilon > 0$ is a relaxation parameter.

Hypotheses of potential f

$$f \text{ is of class } C^2, \tag{1.5}$$

$$f(0) \;=\; 0, \tag{1.6}$$

$$-c_0 \;\le\; F(s) \;\le\; f(s)s \;+\; c_1,\;\; c_0 \ge 0,\;\; c_1 \ge 0,\;\; s \in \mathbb{R}, \tag{1.7}$$

$$\text{with } F(s) \;=\; \int_0^s f(\tau)d\tau,$$

$$|f'(s)| \;\le\; c_2(|s|^{2p} \;+\; 1),\;\; c_2 > 0,\;\; p > 0,\;\; s \in \mathbb{R}, \tag{1.8}$$

$$f'(s) \;\ge\; -c_3,\;\; c_3 \ge 0,\;\; s \in \mathbb{R}. \tag{1.9}$$

Very often, we will need restrictions on p when $n = 3$; these will be precised when needed.

Our aim in this paper is to show the existence of global attractor of the hyperbolic relaxation system (1.1)-(1.4).

Such studies have already been made in many works; in the case of a parabolic-hyperbolic phase-field system (see [2]-[7]). We can also mention the recent work of Daniel Moukoko, for example [7] and [8] in which the hyperbolic system was the subject of a study with regular potential $f(s) \;=\; s^3 - s$, and [9] in which the hyperbolic relaxation system was the subject of a study with a singular potential. In [6] Doumbé Bongola brice Landry has studied the same system as in this article, but the parameter of relaxation $\epsilon = 0$.

2 Notations

*$(.,.)$ denotes the scalar product on $L^2(\Omega)$, and $\|.\|$ the associated norm.
*$(.,.)_X$ denotes the scalar product on X, and $\|.\|_X$ the associated norm.
*$|\Omega|$ is a measure of Ω.
*$H^\kappa(\Omega) = W^{\kappa,2}(\Omega)$ is Sobolev classic space.
*$\varepsilon^\kappa(\epsilon)$ coincide with $[H^{\kappa+1}(\Omega) \times H^\kappa(\Omega)] \cap \{\zeta|_{\partial\Omega} = 0\}$ if $\epsilon > 0$ and with $[H^{\kappa+1}(\Omega) \times H^{\kappa-1}(\Omega)] \cap \{\zeta|_{\partial\Omega} = 0\}$ if $\epsilon = 0$, whenever the traces make sense. Note that when $\kappa = 0$, we write $\varepsilon(\epsilon)$ instead of $\varepsilon^0(\epsilon)$.
*$\|\,.\,\|^2_{\varepsilon^\kappa(\epsilon)}$ with $\epsilon > 0$, is energy norm in $\varepsilon^\kappa(\epsilon)$ for the equation (1.1) with Dirichlet boundary counditions, defined by

$$\|\zeta_v(t)\|^2_{\varepsilon^\kappa(\epsilon)} = \|\left(v(t), \tfrac{\partial v(t)}{\partial t}\right)\|^2_{\varepsilon^\kappa(\epsilon)} = \|v\|^2_{H^{\kappa+1}} + \epsilon\|\tfrac{\partial v}{\partial t}\|^2_{H^\kappa} + \|\tfrac{\partial v}{\partial t}\|^2_{H^{\kappa-1}}.$$

*c_p is the constant of Poincaré.

3 Preliminary Results

We begin by recalling below the two theorems (see [10] Theorem 3.2 and Theorem 3.3) very useful later.

Theorem 3.1 *If* $(u_0,\, u_1,\, \alpha_0,\, \alpha_1) \in H_0^1(\Omega)\times L^2(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)$ *and* $F(u_0) < +\infty$, *then the problem (1.1)-(1.4) has a unique solution* $(u,\, \alpha)$ *such that* $u,\, \alpha \in L^\infty(0,\, T\,; H_0^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,\, T\,; L^2(\Omega)) \cap L^2(0,\, T\,; L^2(\Omega))$ *and* $\frac{\partial \alpha}{\partial t} \in L^\infty(0,\, T\,; L^2(\Omega)) \cap L^2(0,\, T\,; H_0^1(\Omega))$, *for all* $T > 0$.

With more regularity we got the second theorem

Theorem 3.2 *If* $(u_0,\, u_1,\, \alpha_0,\, \alpha_1) \in (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$, *then the problem (1.1)-(1.4) has a unique solution* $(u,\, \alpha)$ such that $u,\, \alpha \in L^\infty(0,\, T\,; H^2(\Omega)\cap H_0^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,\, T\,; H_0^1(\Omega))\cap L^2(0,\, T\,; H_0^1(\Omega))$, $\frac{\partial \alpha}{\partial t} \in L^\infty(0,\, T\,; H_0^1(\Omega))\cap L^2(0,\, T\,; H^2(\Omega)\cap H_0^1(\Omega))$ *and* $\frac{\partial^2 u}{\partial t^2},\, \frac{\partial^2 \alpha}{\partial t^2} \in L^2(0,\, T\,; L^2(\Omega))$, *for all* $T > 0$.

We have thanks to the Theorems 3.1 and 3.2, two respective phase spaces

$$\Phi_0 = \varepsilon(\epsilon)\times H_0^1(\Omega)\times L^2(\Omega)$$

and

$$\Phi_1 = \varepsilon^1(\epsilon)\times \big(H^2(\Omega)\cap H_0^1(\Omega)\big)\times H_0^1(\Omega)$$

and we have two energy norms for the system (1.1)-(1.4), in those phase spaces

$$\|(\zeta_u, \alpha, \tfrac{\partial \alpha}{\partial t})\|^2_{\Phi_\kappa} = \|\zeta_u\|^2_{\varepsilon^\kappa(\epsilon)} + \|\alpha\|^2_{H^{\kappa+1}} + \|\tfrac{\partial \alpha}{\partial t}\|^2_{H^\kappa} \text{ for } \kappa = 0, 1.$$

We then define the continuous semigroup,

$$\begin{aligned} S_\epsilon(t) : \Phi_\kappa \quad &\longrightarrow \quad \Phi_\kappa \\ (\zeta_{u_0}, \alpha_0, \alpha_1) \quad &\longmapsto \quad (\zeta_u(t), \alpha(t), \frac{\partial \alpha(t)}{\partial t}), \end{aligned}$$

for $\kappa = 0, 1$, with $(\zeta_u(t), \alpha(t), \frac{\partial \alpha(t)}{\partial t})$ such that (u, α) is the unique solution of problem (1.1)-(1.4) and $(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t}) = (\zeta_{u_0}, \alpha_0, \alpha_1)$.

4 Main Results

In this study, we have two main results; the dissipativity and regularity of the semigroup $\{S_\epsilon(t)\}_{t\geq 0}$ associated to the problem (1.1)-(1.4) and the existence of the global attractor.

5 Dissipativity and Regularity

The dissipativity and regularity of the semigroup $\{S_\epsilon(t)\}_{t\geq 0}$ associated to the problem (1.1)-(1.4) mean that the semigroup $\{S_\epsilon(t)\}_{t\geq 0}$ associated to the problem (1.1)-(1.4), possesses a bounded absorbing set.

The following lemma gives the uniform estimates of $\|u\|_{H^1}$, $\|\alpha\|_{H^1}$ and $\|\frac{\partial \alpha}{\partial t}\|$ which are independent of ϵ.

Lemma 5.1. *Assume the hypotheses of Theorem 3.1 verified,* $\epsilon \leq 1$ *and* $(u,\, \alpha)$ *the solution of problem* (1.1)-(1.4) *such that* $(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t}) \in \Phi_0$. *Then, the solution* $(u,\, \alpha)$ *satisfies the following estimate*

$$\begin{aligned} &\|u(t)\|^2_{H^1} + \epsilon\|\frac{\partial u(t)}{\partial t}\|^2 + \|\alpha(t)\|^2_{H^1} + \|\frac{\partial \alpha(t)}{\partial t}\|^2 \\ &+ \int_0^t (\|\frac{\partial u(\tau)}{\partial t}\|^2 + \|\frac{\partial \alpha(\tau)}{\partial t}\|^2_{H^1}) e^{-\beta(t-\tau)} d\tau \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0}) e^{-\beta t} + C, \end{aligned} \tag{5.1}$$

where the positive constants β, C *and the monotonic function* Q *are independent of* ϵ.

Proof. We multiply (1.1) by u and (1.2) by α and we have, integrating over Ω, thanks to (1.7)

$$\frac{d}{dt}\left(\|u\|^2+2\epsilon(\frac{\partial u}{\partial t},u)\right)+2\int_\Omega F(u)dx+\|u\|_{H^1}^2 \leq c_p\|\frac{\partial \alpha}{\partial t}\|^2+2\|\frac{\partial u}{\partial t}\|^2+2c_1|\Omega|, \quad (5.2)$$

$$\frac{d}{dt}\left(\|\alpha\|^2+\|\alpha\|_{H^1}^2+2(\frac{\partial \alpha}{\partial t},\alpha)\right)+\|\alpha\|_{H^1}^2 \leq 2c_p^2\|u\|_{H^1}^2+2c_p\|\frac{\partial u}{\partial t}\|^2+2\|\frac{\partial \alpha}{\partial t}\|^2. \quad (5.3)$$

We multiply (1.1) by $\frac{\partial u}{\partial t}$ and (1.2) by $\frac{\partial \alpha}{\partial t}$, we integrate over Ω and we have, summing the two resulting differential equalities, the following estimate

$$\frac{d}{dt}\left(\|u\|_{H^1}^2+\epsilon\|\frac{\partial u}{\partial t}\|^2+2\int_\Omega F(u)dx+\|\alpha\|_{H^1}^2+\|\frac{\partial \alpha}{\partial t}\|^2\right)+2\|\frac{\partial u}{\partial t}\|^2$$
$$+\|\frac{\partial \alpha}{\partial t}\|^2+2\|\frac{\partial \alpha}{\partial t}\|_{H^1}^2 \leq c_p\|u\|_{H^1}^2. \quad (5.4)$$

Summing $\gamma_1(5.2)$, $\gamma_2(5.3)$ and $\gamma_3(5.4)$, where γ_1, γ_2 and $\gamma_3>0$ are such that

$$\begin{aligned}\gamma_1-2c_p^2\gamma_2-c_p\gamma_3 &> 0\\ 2\gamma_3-2\gamma_1-2c_p\gamma_2 &> 0,\\ \gamma_3-c_p\gamma_1-2\gamma_2 &> 0\end{aligned}$$

we find

$$\frac{d}{dt}E_3+C_1\|u\|_{H^1}^2+C_2\|\frac{\partial u}{\partial t}\|^2+2\gamma_1\int_\Omega F(u)dx+\gamma_2\|\alpha\|_{H^1}^2+C_3\|\frac{\partial \alpha}{\partial t}\|^2+2\|\frac{\partial \alpha}{\partial t}\|_{H^1}^2 \leq C \quad (5.5)$$

where the positive constants C_i and C are independent of ϵ, and

$$E_3(t) = \gamma_1\left(\|u(t)\|^2+2\epsilon(\frac{\partial u(t)}{\partial t},u(t))\right)+\gamma_2\left(\|\alpha(t)\|^2+\|\alpha(t)\|_{H^1}^2+2(\frac{\partial \alpha(t)}{\partial t},\alpha(t))\right)$$
$$+\gamma_3\left(\|u(t)\|_{H^1}^2+\epsilon\|\frac{\partial u(t)}{\partial t}\|^2+2\int_\Omega F(u(t))dx+\|\alpha(t)\|_{H^1}^2+\|\frac{\partial \alpha(t)}{\partial t}\|^2\right).$$

Moreover, for sufficiently small values of $\gamma_1>0$ and $\gamma_2>0$, there exists $C>0$ independent of ϵ such that

$$C^{-1}(\|u(t)\|_{H^1}^2+\epsilon\|\frac{\partial u(t)}{\partial t}\|^2+\|\alpha(t)\|_{H^1}^2+\|\frac{\partial \alpha(t)}{\partial t}\|^2)\leq E_3(t)$$
$$\leq C(\|u(t)\|_{H^1}^2+\epsilon\|\frac{\partial u(t)}{\partial t}\|^2+\|\alpha(t)\|_{H^1}^2+\|\frac{\partial \alpha(t)}{\partial t}\|^2).$$

Owing to the above estimate, (5.5) can be rewritten as

$$\frac{d}{dt}E_3+\beta E_3+C_1\|\frac{\partial u}{\partial t}\|^2+C_2\|\frac{\partial \alpha}{\partial t}\|_{H^1}^2\leq C,$$

where the positive constants β, C_i and C are independent of ϵ.

Applying Gronwall's Lemma, we have

$$\|u(t)\|_{H^1}^2+\epsilon\|\frac{\partial u(t)}{\partial t}\|^2+\|\alpha(t)\|_{H^1}^2+\|\frac{\partial \alpha(t)}{\partial t}\|^2$$
$$+\int_0^t(\|\frac{\partial u(\tau)}{\partial t}\|^2+\|\frac{\partial \alpha(\tau)}{\partial t}\|_{H^1}^2)e^{-\beta(t-\tau)}d\tau\leq Q(\|(\zeta_u(0),\alpha(0),\frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta t}+C,$$

where the positve constants β, C and the monotonic function Q are independent of ϵ. This gives the uniform estimates of $\|u\|_{H^1}$, $\|\alpha\|_{H^1}$ and $\|\frac{\partial \alpha}{\partial t}\|$ which are independent of ϵ. □

Theorem 5.1. *Assume the hypotheses of Theorem 3.1 verified, $\epsilon \leq 1$, and (u, α) the solution of problem (1.1)-(1.4) such that $(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t}) \in \Phi_0$. Then, the solution (u, α) satisfies the following estimate*

$$\|(\zeta_u(t), \alpha(t), \frac{\partial \alpha(t)}{\partial t})\|^2_{\Phi_0}$$
$$+\int_0^t (\|\frac{\partial u(\tau)}{\partial t}\|^2 + \|\frac{\partial \alpha(\tau)}{\partial t}\|^2_{H^1})e^{-\beta(t-\tau)}d\tau \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta t} + C,$$

where the positive constants β, C and the monotonic function Q are independent of ϵ.

Proof. Firstly, we determine the uniform energy estimate of the perturbed damped hyperbolic equation (1.1) with initial conditions and homogenous conditions Dirichlet on the boundary. Equation (1.1) can be rewritten as follows

$$\epsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u = -f(u(t)) + \frac{\partial \alpha(t)}{\partial t} = h_{u,\alpha}(t), \quad \zeta_u|_{t=0} = \zeta^0, \quad u(t)|_{\partial\Omega} = 0. \tag{5.6}$$

Owing to the proposition A.2 (see [7]) found from equation (5.6), we have

$$\begin{aligned} &\|(\zeta_u(t)\|^2_{\varepsilon(\epsilon)} + \int_0^t \|\frac{\partial u(\tau)}{\partial t}\|^2 e^{-\beta(t-\tau)}d\tau \\ &\leq Ce^{-\beta t}(\|(\zeta_u(0)\|^2_{\varepsilon(\epsilon)} + \|h_{u,\alpha}(0)\|^2_{H^{-1}}) \\ &\quad +C\int_0^t (\|h_{u,\alpha}(\tau)\|^2_{H^{-1}} + \|\frac{\partial h_{u,\alpha}(\tau)}{\partial t}\|^2_{H^{-1}})e^{-\beta(t-\tau)}d\tau, \end{aligned} \tag{5.7}$$

where the positive constants β and C are independent of ϵ.

To estimate the last term of the second member about (5.7), we first find the estimate of the term $\|h_{u,\alpha}(\tau)\|^2_{H^{-1}} + \|\frac{\partial h_{u,\alpha}(\tau)}{\partial t}\|^2_{H^{-1}}$. We have

$$\begin{aligned} \|h_{u,\alpha}(\tau)\|^2_{H^{-1}} + \|\frac{\partial h_{u,\alpha}(\tau)}{\partial t}\|^2_{H^{-1}} \leq \|f(u(\tau))\|^2_{H^{-1}} + \|\frac{\partial \alpha(\tau)}{\partial t}\|^2_{H^{-1}} + \|f'(u(\tau))\frac{\partial u(\tau)}{\partial t}\|^2_{H^{-1}} \\ +\|\frac{\partial^2 \alpha(\tau)}{\partial t^2}\|^2_{H^{-1}}. \end{aligned} \tag{5.8}$$

The estimate (5.1) gives uniform estimates of $\|u\|_{H^1}$, $\|\alpha\|_{H^1}$ and $\|\dfrac{\partial \alpha}{\partial t}\|$ independent of ϵ, these imply

$$\|u\|^2_{H^{-1}} \quad \leq \quad C\|u\|^2_{H^1} \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta t} + C \tag{5.9}$$

$$\|\frac{\partial \alpha}{\partial t}\|^2_{H^{-1}} \quad \leq \quad C\|\frac{\partial \alpha}{\partial t}\|^2 \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta t} + C, \tag{5.10}$$

where the positive constants β, C and the monotonic function Q are independent of ϵ.

We get thanks to the hypothesis (1.8),

$$|f(u)| \leq \; c_2 \left(\int_0^{|u|} |s|^{2p} ds + \int_0^{|u|} ds\right) = C\left(|u|^{2p+1} + |u|\right), \tag{5.11}$$

the same hypothesis implies for all $w \in H^1_0(\Omega)$,

$$\begin{aligned} |\,(f(u), w)\,| \quad &\leq \quad C\left(\int_\Omega |u|^{2p}|u||w|dx + \int_\Omega |u||w|dx\right) \\ &\leq \quad C\left(\int_\Omega |u|^{2p}|u||w|dx + \|u\|_{H^1}\|w\|_{H^1}\right). \end{aligned}$$

Furthermore, if $n = 2$ and $p > 0$, we have, owing to Hölder's inequality and the continuous embeding of $H^1(\Omega)$ in $L^4(\Omega)$ and in $L^{4p}(\Omega)$,

$$\begin{aligned} |\,(f(u), w)\,| &\leq C\left(\|u\|_{L^{4p}}^{2p}\|u\|_{L^4}\|w\|_{L^4} + \|u\|_{H^1}\|w\|_{H^1}\right) \\ &\leq C(\|u\|_{H^1}^{2p} + 1)\|u\|_{H^1}\|w\|_{H^1}, \end{aligned} \tag{5.12}$$

and if $n = 3$ and $p \leq 1$, we have, using Hölder's inequality (for $p = 1$) and the continuous embeding of $H^1(\Omega)$ in $L^4(\Omega)$,

$$\begin{aligned} |\,(f(u), w)\,| &\leq C\left(\|u\|_{L^4}^{2}\|u\|_{L^4}\|w\|_{L^4} + \|u\|_{H^1}\|w\|_{H^1}\right) \\ &\leq C(\|u\|_{H^1}^{2} + 1)\|u\|_{H^1}\|w\|_{H^1}. \end{aligned} \tag{5.13}$$

The estimates (5.12) and (5.13) allow to deduce

$$\|f(u)\|_{H^{-1}}^2 \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial\alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta t} + C, \tag{5.14}$$

where the positive constants β, C and the monotonic function Q are independent of ϵ.

We have thanks to the hypothesis (1.8) and for all $w \in H_0^1(\Omega)$, the following estimate

$$\begin{aligned} |(f'(u)\frac{\partial u}{\partial t}, w)| &\leq c_2\left(\int_\Omega |u|^{2p}|\frac{\partial u}{\partial t}||w|dx + \int_\Omega |\frac{\partial u}{\partial t}||w|dx\right) \\ &\leq C\left(\int_\Omega |u|^{2p}|\frac{\partial u}{\partial t}||w|dx + \|\frac{\partial u}{\partial t}\|\|w\|_{H^1}\right). \end{aligned}$$

If $n = 2$ and $p > 0$, we find, using Hölder's inequality, and the continuous embeding of $H^1(\Omega)$ in $L^{2(2p+1)}(\Omega)$,

$$\begin{aligned} |(f'(u)\frac{\partial u}{\partial t}, w)| &\leq C\left(\|u\|_{L^{2(2p+1)}}^{2p}\|\frac{\partial u}{\partial t}\|\|w\|_{L^{2(2p+1)}} + \|\frac{\partial u}{\partial t}\|\|w\|_{H^1}\right) \\ &\leq C(\|u\|_{H^1}^{2p} + 1)\|\frac{\partial u}{\partial t}\|\|w\|_{H^1}. \end{aligned} \tag{5.15}$$

If $n = 3$ and $p \leq 1$, we obtain owing to Hölder's inequality (for $p = 1$) and the continuous embeding of $H^1(\Omega)$ in $L^6(\Omega)$,

$$|(f'(u)\frac{\partial u}{\partial t}, w)| \leq C\left(\|u\|_{L^6}^2\|\frac{\partial u}{\partial t}\|\|w\|_{L^6} + \|\frac{\partial u}{\partial t}\|\|w\|_{H^1}\right) \leq C(\|u\|_{H^1}^2 + 1)\|\frac{\partial u}{\partial t}\|\|w\|_{H^1}. \tag{5.16}$$

The estimates (5.15) and (5.16) allow to deduce

$$\|f'(u)\frac{\partial u}{\partial t}\|_{H^{-1}}^2 \leq C\|\frac{\partial u}{\partial t}\|^2, \tag{5.17}$$

where the positive constant C is independente of ϵ.

Equation (1.2) implies

$$\frac{\partial^2\alpha}{\partial t^2} = -\frac{\partial\alpha}{\partial t} + \Delta\frac{\partial\alpha}{\partial t} + \Delta\alpha - u - \frac{\partial u}{\partial t}.$$

We deduce from the above equation, estimates (5.9), (5.10) and from the uniform estimates of $\|\alpha\|_{H^1}^2$, $\|u\|_{H^1}^2$ and $\|\frac{\partial\alpha}{\partial t}\|_{H^1}^2$, the following estimate

$$\begin{aligned} \|\frac{\partial^2\alpha}{\partial t^2}\|_{H^{-1}}^2 &\leq C\left(\|\frac{\partial\alpha}{\partial t}\|_{H^{-1}}^2 + \|\frac{\partial\alpha}{\partial t}\|_{H^1}^2 + \|\alpha\|_{H^1}^2 + \|u\|_{H^{-1}}^2 + \|\frac{\partial u}{\partial t}\|_{H^{-1}}^2\right) \\ &\leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial\alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta t} + C\left(1 + \|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial\alpha}{\partial t}\|_{H^1}^2\right). \end{aligned} \tag{5.18}$$

The estimates (5.10), (5.14), (5.17) and (5.18) inserted into (5.8) allow to obtain

$$\|h_{u,\alpha}(\tau)\|_{H^{-1}}^2 + \|\frac{\partial h_{u,\alpha}(\tau)}{\partial t}\|_{H^{-1}}^2 \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta t} + C\left(1 + \|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^1}^2\right), \quad (5.19)$$

where the positive constants β, C and the monotonic function Q are independente of ϵ.

We insert (5.19) into (5.7). We find from the estimate (5.1)

$$\begin{aligned}
&\|(\zeta_u(t)\|_{\varepsilon(\epsilon)}^2 + \int_0^t \|\frac{\partial u(\tau)}{\partial t}\|^2 e^{-\beta(t-\tau)} d\tau \\
&\leq Ce^{-\beta t} + C\left(\int_0^t \left(Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0}) + 1\right) e^{-\beta(t-\tau)} d\tau\right) \\
&\quad + C\int_0^t \left(\|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^1}^2\right) e^{-\beta(t-\tau)} d\tau \\
&\leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta t} + C, \quad (5.20)
\end{aligned}$$

where the positive constants β, C and the monotonic function Q are independent of ϵ, but C depends on the initial conditions.

Combining (5.1) and (5.20), we have

$$\begin{aligned}
&\|(\zeta_u(t), \alpha(t), \frac{\partial \alpha(t)}{\partial t})\|_{\Phi_0}^2 \\
&+ \int_0^t (\|\frac{\partial u(\tau)}{\partial t}\|^2 + \|\frac{\partial \alpha(\tau)}{\partial t}\|_{H^1}^2) e^{-\beta(t-\tau)} d\tau \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta t} + C. \quad (5.21)
\end{aligned}$$

□

Corollary 5.1. *The semigroup of operators $S_\epsilon(t), t \geq 0$ associated to the problem (1.1)-(1.4) is dissipative in Φ_0, that's to say, it possesses a bounded absorbing set in Φ_0.*

This corollary is a straightforward consequence of Theorem 5.1.

We denote $B_{R_0}(\epsilon) = \left\{(\zeta_u, \alpha, \frac{\partial \alpha}{\partial t}) \in \Phi_0 / \|(\zeta_u, \alpha, \frac{\partial \alpha}{\partial t})\|_{\Phi_0} \leq R_0\right\}$ where R_0 is large enough, a bounded absorbing set for the semigroup $S_\epsilon(t)$ in Φ_0.

Lemma 5.2. *Assume that the hypotheses of Theorem 3.2 hold, $\epsilon \leq 1$, and (u, α) the solution of problem (1.1)-(1.4) such that $(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t}) \in B_{R_0}(\epsilon) \cap \Phi_1$. Then the solution (u, α) satisfies the following estimate*

$$\begin{aligned}
&\|u(t)\|_{H^2}^2 + \epsilon\|\frac{\partial u(t)}{\partial t}\|_{H^1}^2 + \|\alpha(t)\|_{H^2}^2 + \|\frac{\partial \alpha(t)}{\partial t}\|_{H^1}^2 \\
&+ \int_0^t (\|\frac{\partial u(\tau)}{\partial t}\|_{H^1}^2) e^{-\beta(t-\tau)} d\tau \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_1})e^{-\beta t} + C, \quad (5.22)
\end{aligned}$$

where the positive constants β, C and the monotonic function Q are independent of ϵ.

Proof. We multiply (1.1) by $-\Delta u$ and we get, integrating over Ω

$$\frac{d}{dt}\left(\|u\|_{H^1}^2 + 2\epsilon(\nabla\frac{\partial u}{\partial t}, \nabla u)\right) + 2\|u\|_{H^2}^2 = 2\,(f(u), \Delta u) + 2(\frac{\partial \alpha}{\partial t}, \Delta u) + 2\epsilon\|\frac{\partial u}{\partial t}\|_{H^1}^2, \quad (5.23)$$

The estimate (5.11) allows to obtain owing to the Hölder's inequality, the continuous embeding of $H^1(\Omega)$ in $L^{2(2p+1)}(\Omega)$ and if $n=2$ and $p>0$, the following estimate

$$\begin{aligned}(f(u),\Delta u) &\leq C\left(\|u\|^{2p}_{L^{2(2p+1)}}\|u\|_{L^{2(2p+1)}}\|\Delta u\|+\|u\|\|\Delta u\|\right)\\ &\leq C(\|u\|^{2p}_{H^1}+1)\|u\|_{H^1}\|u\|_{H^2},\end{aligned}\tag{5.24}$$

where $C>0$ is independent of ϵ. Moreover if $n=3$, $p\leq 1$ (for $p=1$) and taking into account the continuous embeding of $H^1(\Omega)$ in $L^6(\Omega)$, we obtain

$$(f(u),\Delta u)\leq C\|u\|^3_{L^6}\|\Delta u\|+C\|u\|\|\Delta u\|\leq C(\|u\|^2_{H^1}+1)\|u\|_{H^1}\|u\|_{H^2},\tag{5.25}$$

where $C>0$ is independent of ϵ.

We deduce thanks to the estimates (5.24) and (5.25), applying young's inequality and considering $(\zeta_u,\alpha,\frac{\partial\alpha}{\partial t})\in B_{R_0}(\epsilon)$

$$(f(u),\Delta u)\quad\leq\quad C(\|u\|^q_{H^1}+1)\|u\|_{H^1}\|u\|_{H^2}\leq C\|u\|_{H^1}\|u\|_{H^2}\leq\frac{1}{2}\|u\|^2_{H^2}+C,\tag{5.26}$$

where $C>0$ is independent of ϵ.

We insert (5.26) into (5.23). We get thanks to (5.1), the following estimate

$$\frac{d}{dt}\left(\|u\|^2_{H^1}+2\epsilon(\nabla\frac{\partial u}{\partial t},\nabla u)\right)+\|u\|^2_{H^2}\quad\leq\quad 2c_p\|\frac{\partial\alpha}{\partial t}\|^2_{H^1}+2\|\frac{\partial u}{\partial t}\|^2_{H^1}+C.\tag{5.27}$$

where $C>0$ is independent of ϵ.

We multiply (1.2) by $-\Delta\alpha$ and we get, integrating over Ω

$$\begin{aligned}\frac{d}{dt}\left(\|\alpha\|^2_{H^1}+\|\alpha\|^2_{H^2}+2(\nabla\frac{\partial\alpha}{\partial t},\nabla\alpha)\right)+\|\alpha\|^2_{H^2} &\leq 2c_p\|u\|^2_{H^1}+2c_p\|\frac{\partial u}{\partial t}\|^2_{H^1}+2\|\frac{\partial\alpha}{\partial t}\|^2_{H^1}\\ &\leq C''\|u\|^2_{H^2}+2c_p\|\frac{\partial u}{\partial t}\|^2_{H^1}+2\|\frac{\partial\alpha}{\partial t}\|^2_{H^1}\end{aligned}\tag{5.28}$$

We multiply (1.1) by $-\Delta\frac{\partial u}{\partial t}$ and (1.2) by $-\Delta\frac{\partial\alpha}{\partial t}$ and we integrate over Ω. We obtain, summing the two defferential resulting equalities

$$\begin{aligned}&\frac{d}{dt}\left(\|u\|^2_{H^2}+\epsilon\|\frac{\partial u}{\partial t}\|^2_{H^1}+\|\alpha\|^2_{H^2}+\|\frac{\partial\alpha}{\partial t}\|^2_{H^1}\right)+2\|\frac{\partial u}{\partial t}\|^2_{H^1}+2\|\frac{\partial\alpha}{\partial t}\|^2_{H^1}+2\|\frac{\partial\alpha}{\partial t}\|^2_{H^2}\\ &\leq 2|(f'(u)\nabla u,\nabla\frac{\partial u}{\partial t})|+2|(\nabla u,\nabla\frac{\partial\alpha}{\partial t})|.\end{aligned}\tag{5.29}$$

The assumption (1.8) allows to find owing to the Hölder's inequality, considering the continuous embeding of $H^1(\Omega)$ in $L^{2(2p+1)}(\Omega)$ and if $n=2$ and $p>0$, the following estimate

$$\begin{aligned}|(f'(u)\nabla u,\nabla\frac{\partial u}{\partial t})| &\leq C\|u\|^{2p}_{L^{2(2p+1)}}\|\nabla u\|_{L^{2(2p+1)}}\|\nabla\frac{\partial u}{\partial t}\|+C\|\nabla u\|\|\nabla\frac{\partial u}{\partial t}\|\\ &\leq C\|u\|^{2p}_{H^1}(\|u\|_{H^2}+1)\|\frac{\partial u}{\partial t}\|_{H^1},\end{aligned}\tag{5.30}$$

where $C>0$ is independent of ϵ. Moreover if $n=3$, $p\leq 1$, taking into account the continuous embeding of $H^1(\Omega)$ in $L^6(\Omega)$ (for $p=1$), we obtain

$$\begin{aligned}|(f'(u)\nabla u,\nabla\frac{\partial u}{\partial t})| &\leq C\|u\|^2_{L^6}\|\nabla u\|_{L^6}\|\nabla\frac{\partial u}{\partial t}\|+C\|\nabla u\|\|\nabla\frac{\partial u}{\partial t}\|\\ &\leq C\|u\|^2_{H^1}(\|u\|_{H^2}+1)\|\frac{\partial u}{\partial t}\|_{H^1},\end{aligned}\tag{5.31}$$

where $C > 0$ is independent of ϵ.

The estimates (5.30) and (5.31) allow to obtain

$$|(f'(u)\nabla u, \nabla \frac{\partial u}{\partial t})| \quad \leq \quad C\|u\|_{H^1}^q(\|u\|_{H^2}+1)\|\frac{\partial u}{\partial t}\|_{H^1} \leq C(\|u\|_{H^2}+1)\|\frac{\partial u}{\partial t}\|_{H^1}, \tag{5.32}$$

where $C > 0$ is independent of ϵ.

We insert (5.32) into (5.29). We find

$$\frac{d}{dt}\left(\|u\|_{H^2}^2 + \epsilon\|\frac{\partial u}{\partial t}\|_{H^1}^2 + \|\alpha\|_{H^2}^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^1}^2\right) + \|\frac{\partial u}{\partial t}\|_{H^1}^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^1}^2 \leq C'''\|u\|_{H^2}^2 + C, \tag{5.33}$$

where the positive constants C''' and C are independent of ϵ.

Summing γ_4(5.27), γ_5(5.28), γ_6(5.33) where γ_4, γ_5 and $\gamma_6 > 0$ are such that

$$\begin{aligned}
\gamma_4 - C''\gamma_4 - 2C'''\gamma_6 &> 0 \\
\gamma_6 - 2\gamma_4 - 2c_p\gamma_5 &> 0, \\
\gamma_6 - 2c_p\gamma_4 - 2\gamma_5 &> 0
\end{aligned}$$

we have

$$\frac{d}{dt}E_4 + C_1\|u\|_{H^2}^2 + C_2\|\frac{\partial u}{\partial t}\|_{H^1}^2 + C_3\|\alpha\|_{H^2}^2 + C_4\|\frac{\partial \alpha}{\partial t}\|_{H^1}^2 \leq C,\, C_i,\, C > 0 \tag{5.34}$$

where

$$\begin{aligned}
E_4 \quad = \quad & \gamma_4(\|\alpha\|_{H^1}^2 + \|\alpha\|_{H^2}^2 + 2(\nabla\frac{\partial \alpha}{\partial t}, \nabla\alpha)) + \gamma_5(\|u\|_{H^1}^2 + 2\epsilon(\nabla\frac{\partial u}{\partial t}, \nabla u)) \\
& + \gamma_6(\|u\|_{H^2}^2 + \epsilon\|\frac{\partial u}{\partial t}\|_{H^1}^2 + \|\alpha\|_{H^2}^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^1}^2),
\end{aligned}$$

for sufficiently small values of $\gamma_4 > 0$ and $\gamma_5 > 0$, there exists $C > 0$ independent of ϵ such that

$$\begin{aligned}
& C^{-1}(\|u(t)\|_{H^2}^2 + \epsilon\|\frac{\partial u(t)}{\partial t}\|_{H^1}^2 + \|\alpha(t)\|_{H^2}^2 + \|\frac{\partial \alpha(t)}{\partial t}\|_{H^1}^2) \leq E_4(t) \\
& \leq C(\|u(t)\|_{H^2}^2 + \epsilon\|\frac{\partial u(t)}{\partial t}\|_{H^1}^2 + \|\alpha(t)\|_{H^2}^2 + \|\frac{\partial \alpha(t)}{\partial t}\|_{H^1}^2).
\end{aligned}$$

We deduce from the above astimate and (5.34) the following estimate

$$\frac{d}{dt}E_4 + \beta E_4 + C\|\frac{\partial u}{\partial t}\|_{H^1}^2 \leq C,$$

where β and C are positive constants independent of ϵ.

Applying Gronwall's lemma, we obtain

$$\begin{aligned}
& \|u(t)\|_{H^2}^2 + \epsilon\|\frac{\partial u(t)}{\partial t}\|_{H^1}^2 + \|\alpha(t)\|_{H^2}^2 + \|\frac{\partial \alpha(t)}{\partial t}\|_{H^1}^2 \\
& + \int_0^t \|\frac{\partial u(\tau)}{\partial t}\|_{H^1}^2 e^{-\beta(t-\tau)}d\tau \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_1})e^{-\beta t} + C,
\end{aligned}$$

where the positve constants β, C and the monotonic function Q are independent of ϵ. Then we have uniform estimates of $\|u\|_{H^2}$, $\|\alpha\|_{H^2}$ and $\|\frac{\partial \alpha}{\partial t}\|_{H^1}$ independent of ϵ. □

Theorem 5.2. *Assume that the hypotheses of Theorem 3.2 hold, $\epsilon \leq 1$, and (u, α) the solution of problem (1.1)-(1.4) such that $(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t}) \in B_{R_0}(\epsilon) \cap \Phi_1$. Then, the solution (u, α) satisfies the following estimate*

$$\|(\zeta_u(t), \alpha(t), \frac{\partial \alpha(t)}{\partial t})\|^2_{\Phi_1} + \int_0^t \|\frac{\partial u(\tau)}{\partial t}\|^2_{H^1} e^{-\beta(t-\tau)} d\tau \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_1}) e^{-\beta t} + C,$$

where the positive constants β, C and the monotonic function Q are independent of ϵ.

Proof. We first determine the uniform energy estimate of the perturbed damped hyperbolic equation (1.1) with initial condtions and homogenous conditions Dirichlet on the boundary. Equation (1.1) can be rewritten as follows

$$\epsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u = -f(u(t)) + \frac{\partial \alpha(t)}{\partial t} = h_{u,\alpha}(t), \quad \zeta_u|_{t=0} = \zeta^0, \quad u(t)|_{\partial\Omega} = 0. \tag{5.35}$$

Owing to the proposition A.1 (see [7]) found from equation (5.35), we have

$$\begin{aligned} &\|\zeta_u(t)\|^2_{\varepsilon^1(\epsilon)} + \int_0^t \|\frac{\partial u(\tau)}{\partial t}\|^2_{H^1} e^{-\beta(t-\tau)} d\tau \\ &\leq Ce^{-\beta t}(\|(\zeta_u(0)\|^2_{\varepsilon^1(\epsilon)} + \|h_{u,\alpha}(0)\|^2) \\ &\quad + C\int_0^t (\|h_{u,\alpha}(\tau)\|^2_{H^1} + \|\frac{\partial h_{u,\alpha}(\tau)}{\partial t}\|^2_{H^{-1}}) e^{-\beta(t-\tau)} d\tau, \end{aligned} \tag{5.36}$$

where the positive constants β and C are independent of ϵ.

To estimate the last term of the second member, we first find the estimate of $\|h_{u,\alpha}(\tau)\|^2_{H^1} + \|\frac{\partial h_{u,\alpha}(\tau)}{\partial t}\|^2_{H^{-1}}$. We have

$$\begin{aligned} \|h_{u,\alpha}(\tau)\|^2_{H^1} + \|\frac{\partial h_{u,\alpha}(\tau)}{\partial t}\|^2_{H^{-1}} \quad &\leq \quad \|f(u(\tau))\|^2_{H^1} + \|\frac{\partial \alpha(\tau)}{\partial t}\|^2_{H^1} \\ &\quad + \|f'(u(\tau))\frac{\partial u(\tau)}{\partial t}\|^2_{H^{-1}} + \|\frac{\partial^2 \alpha(\tau)}{\partial t^2}\|^2_{H^{-1}}. \end{aligned} \tag{5.37}$$

We also have thanks to the assumption (1.8),

$$\begin{aligned} \|f(u)\|^2_{H^1} = \|\nabla f(u)\|^2 = \|f'(u)\nabla u\|^2 \quad &\leq \quad c_2 \left(\int_\Omega |u|^{4p} |\nabla u|^2 dx + \int_\Omega |\nabla u|^2 dx \right) \\ &\leq \quad c_2 \left(\int_\Omega |u|^{4p} |\nabla u|^2 dx + \|u\|^2_{H^1} \right). \end{aligned} \tag{5.38}$$

If $n = 2$ and $p > 0$, we find, using the Hölder's inequality and owing to the continuous embeding of $H^1(\Omega)$ in $L^{6p}(\Omega)$ and in $L^6(\Omega)$, the estimate

$$\|f(u)\|^2_{H^1} \quad \leq \quad c_2\|u\|^{4p}_{L^{6p}} \|\nabla u\|^2_{L^6} + c_2\|u\|^2_{H^1} \leq C\|u\|^{4p}_{H^1}\|u\|^2_{H^2} + c_2\|u\|^2_{H^1}. \tag{5.39}$$

If $n = 3$ and $p \leq 1$, we find owing to Hölder's inequality (for $p = 1$) and considering the continuous embeding of $H^1(\Omega)$ in $L^6(\Omega)$, the estimate

$$\|f(u)\|^2_{H^1} \quad \leq \quad c_2\|u\|^4_{L^6}\|\nabla u\|^2_{L^6} + c_2\|u\|^2_{H^1} \leq C\|u\|^4_{H^1}\|u\|^2_{H^2} + c_2\|u\|^2_{H^1}. \tag{5.40}$$

We deduce the below astimate, thanks to the estimates (5.39) and (5.40), considering $(\zeta_u, \alpha, \frac{\partial \alpha}{\partial t}) \in B_{R_0}(\epsilon)$ and the uniform estimate of $\|u\|_{H^2}$

$$\|f(u)\|^2_{H^1} \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_1}) e^{-\beta t} + C. \tag{5.41}$$

Equation (1.2) implies

$$\frac{\partial^2\alpha}{\partial t^2} = -\frac{\partial\alpha}{\partial t} + \Delta\frac{\partial\alpha}{\partial t} + \Delta\alpha - u - \frac{\partial u}{\partial t}.$$

Thanks to the above equation and the uniform estimates of $\|u\|_{H^2}$, $\|\alpha\|_{H^2}$ and $\|\frac{\partial\alpha}{\partial t}\|_{H^1}$, we deduce the following estimate

$$\begin{aligned}\|\frac{\partial^2\alpha}{\partial t^2}\|^2_{H^{-1}} &\leq C\left(\|\frac{\partial\alpha}{\partial t}\|^2_{H^{-1}} + \|\frac{\partial\alpha}{\partial t}\|^2_{H^1} + \|\alpha\|^2_{H^1} + \|u\|^2_{H^{-1}} + \|\frac{\partial u}{\partial t}\|^2_{H^{-1}}\right)\\ &\leq C\left(\|\frac{\partial\alpha}{\partial t}\|^2_{H^1} + \|\alpha\|^2_{H^2} + \|u\|^2_{H^2} + \|\frac{\partial u}{\partial t}\|^2_{H^1}\right)\\ &\leq Q(\|(\zeta_u(0),\alpha(0),\frac{\partial\alpha(0)}{\partial t})\|_{\Phi_1})e^{-\beta t} + C(1+\|\frac{\partial u}{\partial t}\|_{H^1}). \end{aligned}\qquad(5.42)$$

The estimates (5.17), (5.41) and (5.42) inserted into (5.37) allow to obtain

$$\|h_{u,\alpha}(\tau)\|^2_{H^1} + \|\frac{\partial h_{u,\alpha(\tau)}}{\partial t}\|^2_{H^{-1}} \leq Q(\|(\zeta_u(0),\alpha(0),\frac{\partial\alpha(0)}{\partial t})\|_{\Phi_1})e^{-\beta t} + C(1+\|\frac{\partial u}{\partial t}\|_{H^1}). \qquad (5.43)$$

We insert (5.43) into (5.36). we find from the estimate (5.22)

$$\begin{aligned}&\|(\zeta_u(t)\|^2_{\varepsilon^1(\epsilon)} + \int_0^t \|\frac{\partial u(\tau)}{\partial t}\|^2_{H^1} e^{-\beta(t-\tau)}d\tau\\ &\leq Ce^{-\beta t} + C\left(\int_0^t \left(Q(\|(\zeta_u(0),\alpha(0),\frac{\partial\alpha(0)}{\partial t})\|_{\Phi_1}) + 1\right)e^{-\beta(t-\tau)}d\tau\right)\\ &\quad + C\int_0^t \|\frac{\partial u(\tau)}{\partial t}\|^2_{H^1} e^{-\beta(t-\tau)}d\tau\\ &\leq Q(\|(\zeta_u(0),\alpha(0),\frac{\partial\alpha(0)}{\partial t})\|_{\Phi_1})e^{-\beta t} + C, \end{aligned}\qquad(5.44)$$

where the positive constants β, C and the monotonic function Q are independent of ϵ, but C depends on the initial conditions.

Combining (5.22) and (5.44) we have

$$\begin{aligned}&\|(\zeta_u(t),\alpha(t),\frac{\partial\alpha(t)}{\partial t})\|^2_{\Phi_1}\\ &+\int_0^t (\|\frac{\partial u(\tau)}{\partial t}\|^2_{H^1} + \|\frac{\partial\alpha(\tau)}{\partial t}\|^2_{H^2})e^{-\beta(t-\tau)}d\tau \leq Q(\|(\zeta_u(0),\alpha(0),\frac{\partial\alpha(0)}{\partial t})\|_{\Phi_1})e^{-\beta t} + C, \end{aligned}\qquad(5.45)$$

where the positive constants β, C and the monotonic function Q are independent of ϵ. □

Corollary 5.2. *The semigroup of operator $S_\epsilon(t)$ associated to the system* (1.1)-(1.2) *is dissipative in Φ_1, ie, it possesses a bounded absorbing set in Φ_1.*

This corollary is a straightfoward consequence of Theorem 5.2.

6 Existence of Global Attractor

Theorem 6.1. *Assume that the hypotheses of Theorem 5.2 hold. Then the semigroup $S_\epsilon(t)$, $t \geq 0$ defined from Φ_0 in Φ_0, and associated to the problem* (1.1)-(1.4) *possesses a global attractor $\mathcal{A}_\epsilon$ which is compact in Φ_0, bounded and connexe in Φ_1.*

Proof. We have already proved the dissipativity and regularity of the semigroup $\{S_\epsilon(t)\}_{t\geq 0}$ associated to the problem (1.1)-(1.4). It remains to split the semigroup $S_\epsilon(t)$ as the sum of two continuous operators $S^1_\epsilon(t)$ and $S^2_\epsilon(t)$, such that the solution $(u,\, \alpha)$ with initial condition belonging to $B_{R_0} \cap \Phi_1$ can be write as follows

$$
\begin{aligned}
(u,\, \alpha) &= (\nu,\, \eta) + (\omega,\, \xi) \;\; with \\
S^1_\epsilon(t)(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t}) &= (\zeta_\nu(t), \eta(t), \frac{\partial \eta(t)}{\partial t}), \\
S^2_\epsilon(t)(0,0,0) &= (\zeta_\omega(t), \xi(t), \frac{\partial \xi(t)}{\partial t}),
\end{aligned}
$$

where $S^1_\epsilon(t)$ is the solving operator associated to the linear hyperbolic system

$$\epsilon \frac{\partial^2 \nu}{\partial t^2} + \frac{\partial \nu}{\partial t} - \Delta \nu = \frac{\partial \eta}{\partial t}, \tag{6.1}$$

$$\frac{\partial^2 \eta}{\partial t^2} + \frac{\partial \eta}{\partial t} - \Delta \frac{\partial \eta}{\partial t} - \Delta \eta = -\, \nu - \frac{\partial \nu}{\partial t}, \tag{6.2}$$

$$
\begin{aligned}
\nu|_{\partial\Omega} &= \eta|_{\partial\Omega} = 0, \\
\nu|_{t=0} &= u_0, \;\; \frac{\partial \nu}{\partial t}|_{t=0} = u_1, \\
\eta|_{t=0} &= \alpha_0, \;\; \frac{\partial \eta}{\partial t}|_{t=0} = \alpha_1,
\end{aligned}
$$

$S^2_\epsilon(t)$ is the solving operator associated to the nonlinear hyperbolic system

$$\epsilon \frac{\partial^2 \omega}{\partial t^2} + \frac{\partial \omega}{\partial t} - \Delta \omega + f(u) = \frac{\partial \xi}{\partial t}, \tag{6.3}$$

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{\partial \xi}{\partial t} - \Delta \frac{\partial \xi}{\partial t} - \Delta \xi = -\, \omega - \frac{\partial \omega}{\partial t}, \tag{6.4}$$

$$
\begin{aligned}
\omega|_{\partial\Omega} &= \xi|_{\partial\Omega} = 0, \\
\omega|_{t=0} = \frac{\partial \omega}{\partial t}|_{t=0} = \xi|_{t=0} &= \frac{\partial \xi}{\partial t}|_{t=0} = 0,
\end{aligned}
$$

and to show that the operator $S^1_\epsilon(t)$ uniformly converges to 0 over all bounded subset of Φ_0 and $S^2_\epsilon(t)$ is regularizing on Φ_1, when the time t tends to the infinity.

We first prove that the operator $S^1_\epsilon(t)$ uniformly converges to 0 over all bounded subset of Φ_0, when the time t tends to the infinity.

We multiply (6.1) by ν and (6.2) by η and we get, integrating over Ω.

$$\frac{d}{dt}\left(\|\nu\|^2 + 2\epsilon(\frac{\partial \nu}{\partial t}, \nu) \right) + \|\nu\|^2_{H^1} \leq c_p \|\frac{\partial \eta}{\partial t}\|^2 + 2\|\frac{\partial \nu}{\partial t}\|^2, \tag{6.5}$$

$$\frac{d}{dt}\left(\|\eta\|^2 + \|\eta\|^2_{H^1} + 2(\frac{\partial \eta}{\partial t}, \eta) \right) + \|\eta\|^2_{H^1} \leq 2c_p^2 \|\nu\|^2_{H^1} + 2c_p \|\frac{\partial \nu}{\partial t}\|^2 + 2\|\frac{\partial \eta}{\partial t}\|^2. \tag{6.6}$$

We multiply (6.1) by $\frac{\partial \nu}{\partial t}$ and (6.2) by $\frac{\partial \eta}{\partial t}$ and we obtain, integrating over Ω, the two following equalities

$$
\begin{aligned}
\frac{d}{dt}\left(\|\nu\|^2_{H^1} + \epsilon \|\frac{\partial \nu}{\partial t}\|^2 \right) + 2\|\frac{\partial \nu}{\partial t}\|^2 &= 2(\frac{\partial \eta}{\partial t},\, \frac{\partial \nu}{\partial t}), \\
\frac{d}{dt}\left(\|\eta\|^2_{H^1} + \|\frac{\partial \eta}{\partial t}\|^2 \right) + 2\|\frac{\partial \eta}{\partial t}\|^2 + 2\|\frac{\partial \eta}{\partial t}\|^2_{H^1} &= -2(\nu, \frac{\partial \eta}{\partial t}) - 2(\frac{\partial \nu}{\partial t},\, \frac{\partial \eta}{\partial t}),
\end{aligned}
$$

the sum of the above equalities allow to obtain the following estimate

$$\frac{d}{dt}\left(\|\nu\|^2_{H^1}+\epsilon\|\frac{\partial\nu}{\partial t}\|^2+\|\eta\|^2_{H^1}\ +\ \|\frac{\partial\eta}{\partial t}\|^2\right)+2\|\frac{\partial\nu}{\partial t}\|^2+\|\frac{\partial\eta}{\partial t}\|^2+\ 2\|\frac{\partial\eta}{\partial t}\|^2_{H^1}\le c_p\|\nu\|^2_{H^1}. \quad (6.7)$$

Summing $\gamma_8(6.5)$, $\gamma_9(6.6)$ and $\gamma_{10}(6.7)$ where γ_8, γ_9 and $\gamma_{10}>0$ are such that

$$\begin{aligned}\gamma_8-2c_p^2\gamma_9 &> 0\\ \gamma_{10}-c_P\gamma_9-\gamma_8 &> 0,\\ \gamma_{10}-c_p\gamma_8-2\gamma_9 &> 0\end{aligned}$$

we get

$$\frac{d}{dt}E_5+C_1\|\nu\|^2_{H^1}+C_2\|\frac{\partial\nu}{\partial t}\|^2+C_3\|\eta\|^2_{H^1}+C_4\|\frac{\partial\eta}{\partial t}\|^2+C_5\|\frac{\partial\eta}{\partial t}\|^2_{H^1}\le 0,\ C_i>0 \quad (6.8)$$

where

$$\begin{aligned}E_5 \quad = \quad &\gamma_8(\|\nu\|^2+2\epsilon(\frac{\partial\nu}{\partial t},\nu))+\gamma_9(\|\eta\|^2+\|\eta\|^2_{H^1}+2(\frac{\partial\eta}{\partial t},\eta))\\ &+\gamma_{10}(\|\nu\|^2_{H^1}+\epsilon\|\frac{\partial\nu}{\partial t}\|^2+\|\eta\|^2_{H^1}\ +\ \|\frac{\partial\eta}{\partial t}\|^2).\end{aligned}$$

Moreover for sufficiently small values of γ_8 and $\gamma_9>0$, there exists $C>0$ independent of ϵ such that

$$\begin{aligned}&C^{-1}(\|\nu(t)\|^2_{H^1}+\epsilon\|\frac{\partial\nu(t)}{\partial t}\|^2+\|\eta(t)\|^2_{H^1}\ +\ \|\frac{\partial\eta(t)}{\partial t}\|^2)\le E_5(t)\\ &\le C(\|\nu(t)\|^2_{H^1}+\epsilon\|\frac{\partial\nu(t)}{\partial t}\|^2+\|\eta(t)\|^2_{H^1}\ +\ \|\frac{\partial\eta(t)}{\partial t}\|^2).\end{aligned}$$

We have the bellow estimate, thanks to (6.8) and the above estimate

$$\frac{d}{dt}E_5+\beta E_5+C_1\|\frac{\partial\nu}{\partial t}\|^2+C_2\|\frac{\partial\eta}{\partial t}\|^2_{H^1}\le 0, \quad (6.9)$$

where the positive constants β, C_1 and C_2 are independent of ϵ.

Applying Gronwall's lemma, we get

$$\begin{aligned}&\|\nu(t)\|^2_{H^1}+\epsilon\|\frac{\partial\nu(t)}{\partial t}\|^2+\|\eta(t)\|^2_{H^1}\ +\ \|\frac{\partial\eta(t)}{\partial t}\|^2\\ &+\int_0^t(\|\frac{\partial\nu(\tau)}{\partial t}\|^2+\|\frac{\partial\eta(\tau)}{\partial t}\|^2_{H^1})e^{-\beta(t-\tau)}d\tau\le Q(\|(\zeta_u(0),\alpha(0),\frac{\partial\alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta t}, \quad (6.10)\end{aligned}$$

where the positive constants β, C and the monotonic function Q are independent of ϵ.

The above estimate allows to obtain the uniform estimates of $\|\nu\|_{H^1}$, $\|\eta\|_{H^1}$ and $\|\frac{\partial\eta}{\partial t}\|$ independent of ϵ.

The uniform energy estimate of the perturbed damped hyperbolic equation (6.1) with initial conditions and homogenous conditions Dirichlet on the boundary, is determined as follows.

We have thanks to the equation (6.1),

$$\epsilon\frac{\partial^2\nu}{\partial t^2}+\frac{\partial\nu}{\partial t}-\Delta\nu=\frac{\partial\eta(t)}{\partial t}=h_\eta(t),\quad \zeta_\nu|_{t=0}=\zeta^0,\quad \nu(t)|_{\partial\Omega}=0. \quad (6.11)$$

The proposition A.2 (see [7]) allows to find from equation (6.11), the estimate

$$\|(\zeta_\nu(t)\|^2_{\varepsilon(\epsilon)} + \int_0^t \|\frac{\partial \nu(\tau)}{\partial t}\|^2 e^{-\beta(t-\tau)} d\tau$$
$$\leq Ce^{-\beta t}(\|(\zeta_\nu(0)\|^2_{\varepsilon(\epsilon)} + \|h_\eta(0)\|^2_{H^{-1}}) + C\int_0^t (\|h_\eta(\tau)\|^2_{H^{-1}} + \|\frac{\partial h_\eta(\tau)}{\partial t}\|^2_{H^{-1}}) e^{-\beta(t-\tau)} d\tau, \quad (6.12)$$

where the positive constants C and β are independent of ϵ.

We first find the estimate of $\|h_\eta(\tau)\|^2_{H^{-1}} + \|\frac{\partial h_\eta(\tau)}{\partial t}\|^2_{H^{-1}}$. We have

$$\|h_\eta(\tau)\|^2_{H^{-1}} + \|\frac{\partial h_\eta(\tau)}{\partial t}\|^2_{H^{-1}} \leq \|\frac{\partial \eta(\tau)}{\partial t}\|^2_{H^{-1}} + \|\frac{\partial^2 \eta(\tau)}{\partial t^2}\|^2_{H^{-1}} \quad (6.13)$$

The estimate (6.10) gives uniform estimates of $\|\nu\|_{H^1}$, $\|\eta\|_{H^1}$ and $\|\frac{\partial \eta}{\partial t}\|$ independent of ϵ, these imply

$$\|\nu\|^2_{H^{-1}} \leq C\|\nu\|^2_{H^1} \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0}) e^{-\beta t} \quad (6.14)$$
$$\|\frac{\partial \eta}{\partial t}\|^2_{H^{-1}} \leq C\|\frac{\partial \eta}{\partial t}\|^2 \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0}) e^{-\beta t}, \quad (6.15)$$

where the positive constants β, C and the monotonic function Q are independent of ϵ.

Equation (6.2) implies

$$\frac{\partial^2 \eta}{\partial t^2} = -\frac{\partial \eta}{\partial t} + \Delta \frac{\partial \eta}{\partial t} + \Delta \eta - \nu - \frac{\partial \nu}{\partial t}.$$

We deduce owing to the above equation, estimates (6.14), (6.15) and the uniform estimate of $\|\eta\|_{H^1}$,the following estimate

$$\|\frac{\partial^2 \eta}{\partial t^2}\|^2_{H^{-1}} \leq C\left(\|\frac{\partial \eta}{\partial t}\|^2_{H^{-1}} + \|\frac{\partial \eta}{\partial t}\|^2_{H^1} + \|\eta\|^2_{H^1} + \|\nu\|^2_{H^{-1}} + \|\frac{\partial \nu}{\partial t}\|^2_{H^{-1}}\right)$$
$$\leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0}) e^{-\beta t} + C\left(\|\frac{\partial \nu}{\partial t}\|^2 + \|\frac{\partial \eta}{\partial t}\|^2_{H^1}\right). \quad (6.16)$$

The estimates (6.15) and (6.16) inserted into (6.13) allow to obtain

$$\|h_\eta(\tau)\|^2_{H^{-1}} + \|\frac{\partial h_\eta(\tau)}{\partial t}\|^2_{H^{-1}} \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \eta(0)}{\partial t})\|_{\Phi_0}) e^{-\beta t} + C\left(\|\frac{\partial \nu}{\partial t}\|^2 + \|\frac{\partial \eta}{\partial t}\|^2_{H^1}\right). \quad (6.17)$$

We insert (6.17) into (6.12). We find from the estimate (6.10)

$$\|(\zeta_\nu(t)\|^2_{\varepsilon(\epsilon)} + \int_0^t \|\frac{\partial \nu(\tau)}{\partial t}\|^2 e^{-\beta(t-\tau)} d\tau \leq Ce^{-\beta t} + Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0}) te^{-\beta t}$$
$$+ \int_0^t \left(\|\frac{\partial \nu(\tau)}{\partial t}\|^2 + \|\frac{\partial \eta(\tau)}{\partial t}\|^2_{H^1}\right) e^{-\beta(t-\tau)} d\tau$$
$$\leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0}) e^{-\beta t}, \quad (6.18)$$

Combining (6.18) and (6.10) we have

$$\|(\zeta_\nu(t), \eta(t), \frac{\partial \eta(t)}{\partial t})\|^2_{\Phi_0}$$
$$+ \int_0^t (\|\frac{\partial \nu(\tau)}{\partial t}\|^2 + \|\frac{\partial \eta(\tau)}{\partial t}\|^2_{H^1}) e^{-\beta(t-\tau)} d\tau \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0}) e^{-\beta t}.$$

So the operator $S^1_\epsilon(t)$ uniformly converges to 0 over all bounded subset of Φ_0 when t tends to the infinity.

It remain to prove that $S^2_\epsilon(t)$ is regularizing on Φ_1, when t tends to the infinity.

We multiply (6.3) by ω and (6.4) by ξ and we have, integrating over Ω

$$\frac{d}{dt}\left(\|\omega\|^2 + 2\epsilon(\omega\,,\,\frac{\partial\omega}{\partial t})\right) + \|\omega\|^2_{H^1} \quad\leq\quad 2c_p\|f(u)\|^2 + 2c_p^2\|\frac{\partial\xi}{\partial t}\|^2_{H^1} + 2c_p\|\frac{\partial\omega}{\partial t}\|^2_{H^1}, \qquad (6.19)$$

$$\frac{d}{dt}\left(\|\xi\|^2 + \|\xi\|^2_{H^1} + 2(\xi\,,\,\frac{\partial\xi}{\partial t})\right) + \|\xi\|^2_{H^1} \quad\leq\quad 2c_p^2\|\omega\|^2_{H^1} + 2c_p^2\|\frac{\partial\omega}{\partial t}\|^2_{H^1} + 2c_p\|\frac{\partial\xi}{\partial t}\|^2_{H^1}. \qquad (6.20)$$

We multiply (6.3) by $-\Delta\frac{\partial\omega}{\partial t}$ and (6.4) by $-\Delta\frac{\partial\xi}{\partial t}$ and we get, integrating over Ω

$$\frac{d}{dt}\left(\|\omega\|^2_{H^2} + \epsilon\|\frac{\partial\omega}{\partial t}\|^2_{H^1}\right) + 2\|\frac{\partial\omega}{\partial t}\|^2_{H^1} \quad=\quad -2(f'(u)\nabla u\,,\,\nabla\frac{\partial\omega}{\partial t}) + 2(\nabla\frac{\partial\xi}{\partial t}\,,\,\nabla\frac{\partial\omega}{\partial t}),$$

$$\frac{d}{dt}\left(\|\xi\|^2_{H^2} + \|\frac{\partial\xi}{\partial t}\|^2_{H^1}\right) + 2\|\frac{\partial\xi}{\partial t}\|^2_{H^1} + 2\|\frac{\partial\xi}{\partial t}\|^2_{H^2} \quad=\quad -2(\nabla\omega\,,\,\nabla\frac{\partial\xi}{\partial t}) - 2(\nabla\frac{\partial\omega}{\partial t}\,,\,\nabla\frac{\partial\xi}{\partial t}).$$

The sum of the above equalities allows to find the following estimate

$$\frac{d}{dt}\left(\|\omega\|^2_{H^2} + \epsilon\|\frac{\partial\omega}{\partial t}\|^2_{H^1} + \|\xi\|^2_{H^2} + \|\frac{\partial\xi}{\partial t}\|^2_{H^1}\right) + \|\frac{\partial\omega}{\partial t}\|^2_{H^1} + \|\frac{\partial\xi}{\partial t}\|^2_{H^1} + 2\|\frac{\partial\xi}{\partial t}\|^2_{H^2}$$
$$\leq \|f'(u)\nabla u\|^2 + \|\omega\|^2_{H^1}. \qquad (6.21)$$

We multiply (6.3) by $\frac{\partial^2\omega}{\partial t^2}$ and We get, integrating over Ω, the following estimate

$$\frac{d}{dt}\left(\|\frac{\partial\omega}{\partial t}\|^2 + 2(\nabla\omega\,,\,\nabla\frac{\partial\omega}{\partial t})\right) + \epsilon\|\frac{\partial^2\omega}{\partial t^2}\|^2 \quad\leq\quad C_1\|f(u)\|^2 + C_2\|\frac{\partial\xi}{\partial t}\|^2_{H^1} + 2\|\frac{\partial\omega}{\partial t}\|^2_{H^1}. \qquad (6.22)$$

Summing γ_{11}(6.19), γ_{12}(6.20), γ_{13}(6.21) and γ_{14}(6.22), where γ_{11}, γ_{12}, γ_{13} and $\gamma_{14} > 0$ are such that

$$\begin{aligned} \gamma_{11} - 2c_p^2\gamma_{12} - \gamma_{13} &> 0 \\ \gamma_{13} - 2c_p\gamma_{11} - 2c_p^2\gamma_{12} - 2\gamma_{14} &> 0, \\ \gamma_{13} - 2c_p^2\gamma_{11} - 2c_p\gamma_{12} - C_2\gamma_{14} &> 0 \end{aligned}$$

we find

$$\frac{d}{dt}E_6 + C_3\|\omega\|^2_{H^1} + C_4\|\frac{\partial\omega}{\partial t}\|^2_{H^1} + C_5\|\xi\|^2_{H^1} + C_6\|\frac{\partial\xi}{\partial t}\|^2_{H^1} + C_7\|\frac{\partial\xi}{\partial t}\|^2_{H^2} + C_8\|\frac{\partial^2\omega}{\partial t^2}\|^2$$
$$\leq \|f'(u)\nabla u\|^2 + C_1\|f(u)\|^2,$$

we deduce the estimate

$$\frac{d}{dt}E_6 \leq C_1\|f'(u)\nabla u\|^2 + C_2\|f(u)\|^2, \qquad (6.23)$$

where

$$E_6 = \gamma_{11}\left(\|\omega\|^2 + 2\epsilon(\omega\,,\,\frac{\partial\omega}{\partial t})\right) + \gamma_{12}\left(\|\xi\|^2 + \|\xi\|^2_{H^1} + 2(\xi\,,\,\frac{\partial\xi}{\partial t})\right)$$
$$+\gamma_{13}\left(\|\omega\|^2_{H^2} + \epsilon\|\frac{\partial\omega}{\partial t}\|^2_{H^1} + \|\xi\|^2_{H^2} + \|\frac{\partial\xi}{\partial t}\|^2_{H^1}\right) + \gamma_{14}\left(\|\frac{\partial\omega}{\partial t}\|^2 + 2(\nabla\omega\,,\,\nabla\frac{\partial\omega}{\partial t})\right),$$

For sufficiently small values of γ_{11}, γ_{12} and $\gamma_{14>0}$, there exists $C > 0$ independent of ϵ such that

$$C\|(\zeta_\omega(t), \xi(t), \frac{\partial \xi(t)}{\partial t})\|^2_{\Phi_1} \leq E_6(t), \tag{6.24}$$

The assumption (1.8) allows to find owing to $(u,\, \alpha) \in B_{R_0} \cap \Phi_1$ that's to say $u \in H^2(\Omega))$ with $H^2(\Omega) \subset L^\infty(\Omega)$, the estimates

$$\begin{aligned}
\|f'(u)\nabla u\|^2 &\leq c_2 \int_\Omega |u|^{4p}|\nabla u|^2 dx + c_2 \int_\Omega |\nabla u|^2 dx \leq C\left(\|u\|^{4p}_{L^\infty(\Omega)}\|u\|^2_{H^1} + \|u\|^2_{H^1}\right) \leq C\|u\|^2_{H^1} \\
\|f(u)\|^2 &\leq C\int_\Omega \left(|u|^{4p}|u|^2 + |u|^2\right) dx \leq C\left(\|u\|^{4p}_{L^\infty(\Omega)}\|u\|^2 + \|u\|^2\right) \leq C\|u\|^2_{H^1}.
\end{aligned}$$

Thanks to the estimate (5.1), the above estimates can be write as follows

$$\|f'(u)\nabla u\|^2 \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta\tau} + C \tag{6.25}$$

$$\|f(u)\|^2 \leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta\tau} + C, \tag{6.26}$$

where the positive constants β, C and the monotonic function Q are independent of ϵ. After inserting (6.25) and (6.26) into (6.23), we find

$$\frac{d}{dt}E_6 \leq Q(\|(\zeta_u(0), \alpha(0), \tfrac{\partial \alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta\tau} + C, \tag{6.27}$$

We obtain, integrating (6.27) from 0 to $t \in [0\,,\, T]$ and combining with (6.24),

$$\begin{aligned}
\|(\zeta_\omega(t), \xi(t), \frac{\partial \xi(t)}{\partial t})\|^2_{\Phi_1} &\leq C\int_0^t \left(Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0})e^{-\beta\tau} + C\right) d\tau \\
&\leq \left(1 - e^{-\beta t}\right) Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0}) + Ct \\
&\leq Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0}) + CT \\
&\leq (1+T)\, Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0}) \\
\|(\zeta_\omega(t), \xi(t), \frac{\partial \xi(t)}{\partial t})\|^2_{\Phi_1} &\leq \left(1+T^2\right) Q(\|(\zeta_u(0), \alpha(0), \frac{\partial \alpha(0)}{\partial t})\|_{\Phi_0}).
\end{aligned} \tag{6.28}$$

The estimate (6.28) allows to assert that the operator $S_\epsilon(t)$ is regularizing in Φ_1, and there exists a bounded and attracting compact set in Φ_1. □

7 Conclusion

The works contained in this manuscript about dynamic system, are very interesting to explain the context of phase transition phenomena, when the solution of the system exists. The existence and unicity of global attractor, associated to the problem $(1.1)-(1.4)$ that we have proved in this paper, allow to assert that the solution of the problem $(1.1)-(1.4)$ studied in [4], belongs to the subset called global attractor, from a certain time.

Competing Interests

Authors have declared that no competing interests exist.

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